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TRANSIENT SIGNAL PROPAGATION IN LOSSY PLASMAS

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Air Force Cambridge Research Laboratories
L. G. Hanscom Field, Massachusetts

25 April 1973

AFCRL-TR-73-0277
25 APRIL 1973
PHYSICAL SCIENCES RESEARCH PAPERS, NO. 546

AIR FORCE CAMBRIDGE RESEARCH LABORATORIES

L. G. HANSCOM FIELD, BEDFORD, MASSACHUSETTS

Transient Signal Propagation in Lossy Plasmas

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AIR FORCE SYSTEMS COMMAND

United States Air Force

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R&D	
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)	
1. ORIGINATING ACTIVITY (Corporate author) Air Force Cambridge Research Laboratories (LZP) L.G. Hanscom Field Bedford, Massachusetts 01730	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP
3. REPORT TITLE TRANSIENT SIGNAL PROPAGATION IN LOSSY PLASMAS	
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Scientific. Interim.	
5. AUTHOR(S) (First name, middle initial, last name) Ronald L. Fante Richard L. Taylor	

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
25 April 1973	38	11

8a. CONTRACT OR GRANT NO. b. PROJECT, TASK, WORK UNIT NOS. 5635-04-01 c. DOD ELEMENT 61102F d. DOD SUBELEMENT 681305	9a. ORIGINATOR'S REPORT NUMBER(S) AFCRL-TR-73-0277 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) PSRP No. 546
10. DISTRIBUTION STATEMENT Approved for public release; distribution unlimited.	
11. SUPPLEMENTARY NOTES TECH, OTHER	12. SPONSORING MILITARY ACTIVITY Air Force Cambridge Research Laboratories (LZP) L.G. Hanscom Field Bedford, Massachusetts 01730

13. ABSTRACT

Using contour integration techniques, we have calculated the transient response of a lossy homogeneous isotropic plasma to a unit step and a step-carrier sine wave. We have found that in the presence of losses the temporal step function response within the plasma does not approach zero for large time as for the collisionless case, but rather approaches unity. The rate at which the transient approaches the asymptotic value of unity is strongly dependent on the losses, being quite rapid for large losses and exceedingly slow for very small losses. This result has an important impact on the propagation of EMP through the plasma sheath around a reentry vehicle, and through the ionized region near a low altitude nuclear fireball.

DD FORM 1473 1 NOV 65

Unclassified
Security Classification

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Transient propagation						
Plasmas						

Unclassified
Security Classification

Abstract

Using contour integration techniques, we have calculated the transient response of a lossy homogeneous isotropic plasma to a unit step and a step-carrier sine wave. We have found that in the presence of losses the temporal step function response within the plasma does not approach zero for large time as for the collisionless case, but rather approaches unity. The rate at which the transient approaches the asymptotic value of unity is strongly dependent on the losses, being quite rapid for large losses and exceedingly slow for very small losses. This result has an important impact on the propagation of EMP through the plasma sheath around a reentry vehicle, and through the ionized region near a low altitude nuclear fireball.

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I. INTRODUCTION

There has been a great deal of research done (Haskell and Case, 1966a, b, 1967; Knop and Cohn, 1963; Knop, 1965; Lighthill, 1965; Felsen, 1969; Antonucci, 1972) on the transient propagation of electromagnetic signals through lossless plasmas, and the results are quite appropriate for the calculation of EMP through the undisturbed ionosphere. For transmission of EMP through the plasma surrounding a reentry vehicle or through the ionized region near a low altitude nuclear fireball, however, the theory is quite inadequate since here the effect of collisional absorption on the transient propagation is quite significant. The effect of collisional losses has been considered previously by Field (1971) using the method of characteristics, but unfortunately he does not present results for the transmitted field in the cases of interest to us. In this report, we will therefore extend the previous theories to include the effect of collisional absorption on the propagation of transients in a cold*, homogeneous isotropic plasma. We shall consider only two types of transient signals: the step-carrier sine wave and the unit step. The transient

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* The cold plasma model is approximately valid in those regions near a nuclear burst where the electron temperature is less than 30,000°K. Also, in applying these results to EMP propagation we must be sufficiently far from the fireball to neglect nonlinear effects.

(Received for publication 24 April 1973)

response within the plasma to the unit step is quite important since the response to an arbitrary EMP can always be synthesized as a superposition of the unit step responses.

As a starting point for our investigation we consider the Maxwell equations for the electric field strength E, and the Langevin equation for the electron velocity v. If we write E in terms of the Laplace transform

$$E(x, t) = \int_B \hat{E}(x, p) e^{pt} \, dp \ , \tag{1}$$

it is readily shown that for $x \geq 0$

$$\hat{E}(x, p) = \hat{E}(x=0, p) e^{-\gamma \frac{x}{c}} \ , \tag{2}$$

where

$$\gamma = \left(\frac{p}{p+\nu_c}\right)^{1/2} (p^2 + \nu_c p + \omega_p^2)^{1/2} \ , \tag{3}$$

ω_p = electron plasma frequency,

ν_c = electron - neutral collision frequency.

We have also assumed that $E(x, t=0)=0$ for all $x>0$. In order to evaluate $E(x, t)$, it is necessary to perform the integration along the Bromwich contour B in Eq. (1). This may be accomplished either by integrating directly along the Bromwich contour, or by deforming that contour into the left-half p-plane. This latter choice leads to integrals which rapidly converge, and will therefore be pursued in this report.

2. BRANCH POINTS AND BRANCH CUTS

The function γ in Eq. (3) can be written in product form as

$$\gamma = \gamma_1 \gamma_2 \gamma_3 \gamma_4 \ , \tag{4}$$

where $\gamma_1 = p^{1/2}$, $\gamma_2 = (p+\nu_c)^{-1/2}$, $\gamma_3 = (p-p_1)^{1/2}$, $\gamma_4 = (p-p_2)^{1/2}$. For $\nu_c < 2\omega_p$ we have

$$p_{\frac{1}{2}} = -\frac{\nu_c}{2} \pm i \left(\omega_p^2 - \nu_c^2/4\right)^{1/2} \ ,$$

while for $\nu_c > 2\omega_p$ we have

$$p_{\substack{1\\2}} = -\frac{\nu_c}{2} \pm (\nu_c^2/4 - \omega_p^2)^{1/2} .$$

The function γ_1 has branch points at $p = 0$ and $p = \infty$. For this function we choose the branch cut shown in Figure 1a. The function γ_2 has branch points at $p = -\nu_c$

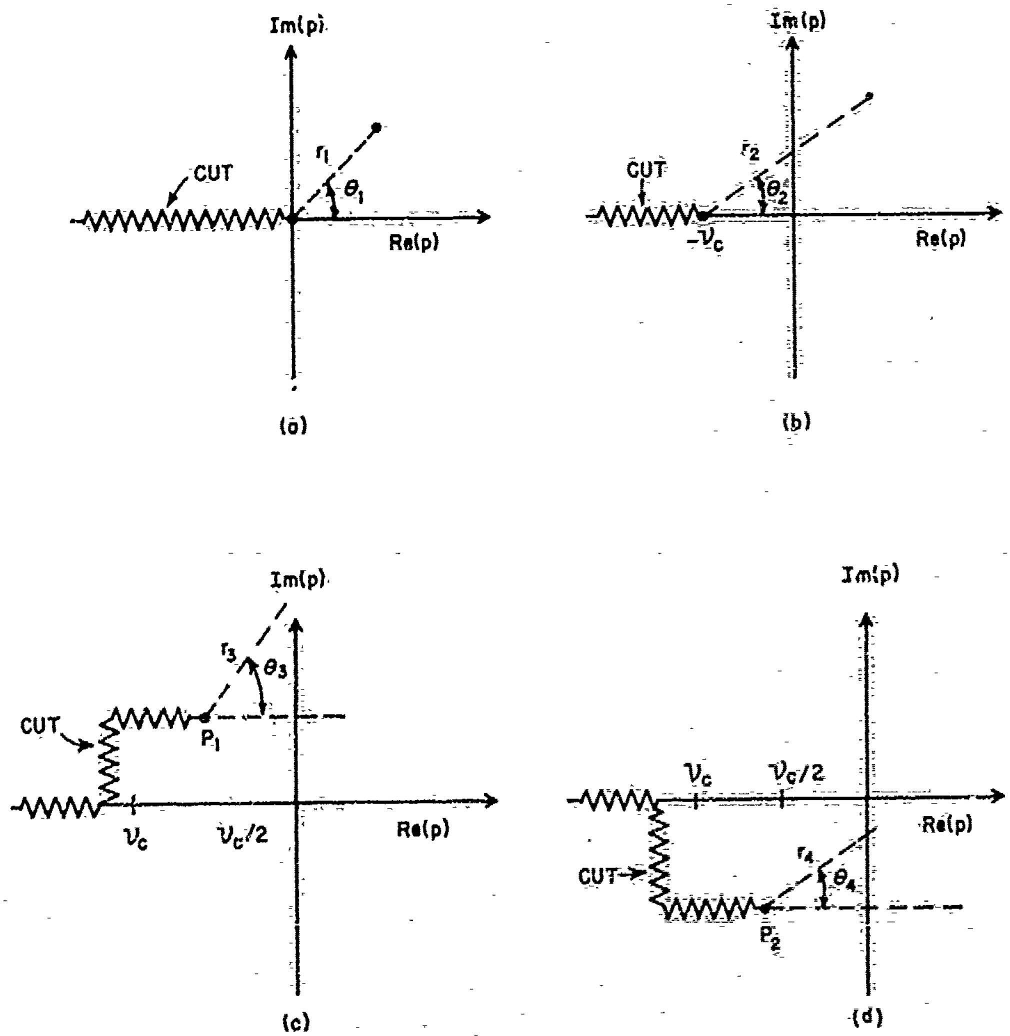

Figure 1. Branch Cuts for the Functions (a) γ_1, (b) γ_2, (c) γ_3, (d) γ_4

and $p = \infty$, and we choose the branch cut shown in Figure 1b for this quantity. For γ_3, which has branch points at $p = p_1$ and $p = \infty$, we choose the branch cut illustrated in Figure 1c, while for γ_4, which has branch points at $p = p_2$ and $p = \infty$, we choose the branch cut shown in Figure 1d. If we write $p = r_1 \exp(i\theta_1)$, $(p+\nu_c) = r_2 \exp(i\theta_2)$, $(p-p_1) = r_3 \exp(i\theta_3)$, $(p-p_2) = r_4 \exp(i\theta_4)$, we can write $\gamma_1 = \pm r_1^{1/2} \exp(i\theta_1/2)$, $\gamma_2 = \pm r_2^{-1/2} \exp(-i\theta_2/2)$, $\gamma_3 = \pm r_3^{1/2} \exp(i\theta_3/2)$ and $\gamma_4 = \pm r_4^{1/2} \exp(i\theta_4/2)$. For each of these functions we then choose the Riemann sheet corresponding to the upper sign, so that

$$\gamma = \left(\frac{r_1 r_3 r_4}{r_2}\right)^{1/2} \exp\left[i\left(\frac{\theta_1-\theta_2+\theta_3+\theta_4}{2}\right)\right], \tag{5}$$

and the branch cuts for the total function $\gamma = \gamma_1\gamma_2\gamma_3\gamma_4$ are those shown in Figure 2 for $\nu_c < 2\omega_p$, and in Figure 3 for $\nu_c > 2\omega_p$. The integral in Eq. (1) can now be performed by deforming the Bromwich Contour B in Figure 2 into the

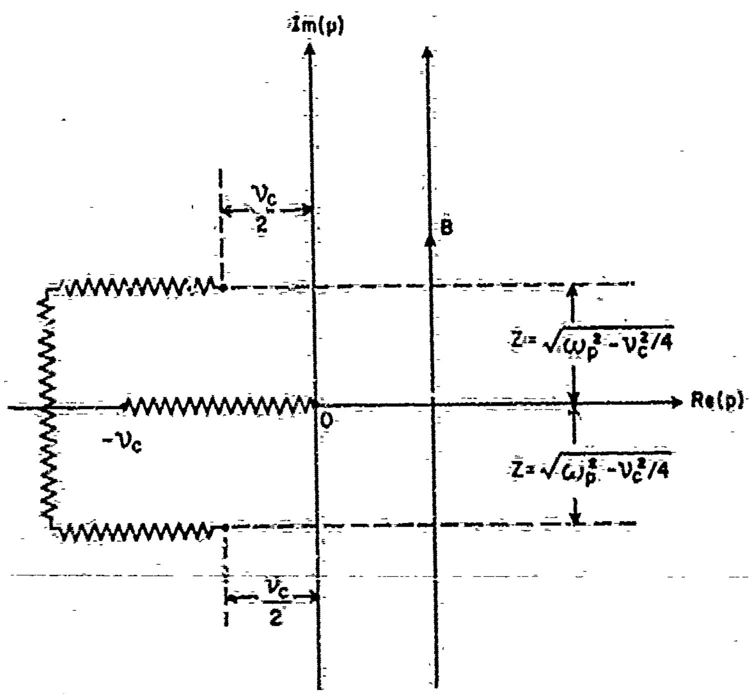

Figure 2. Branch Cuts for the Function γ when $\nu_c < 2\omega_p$

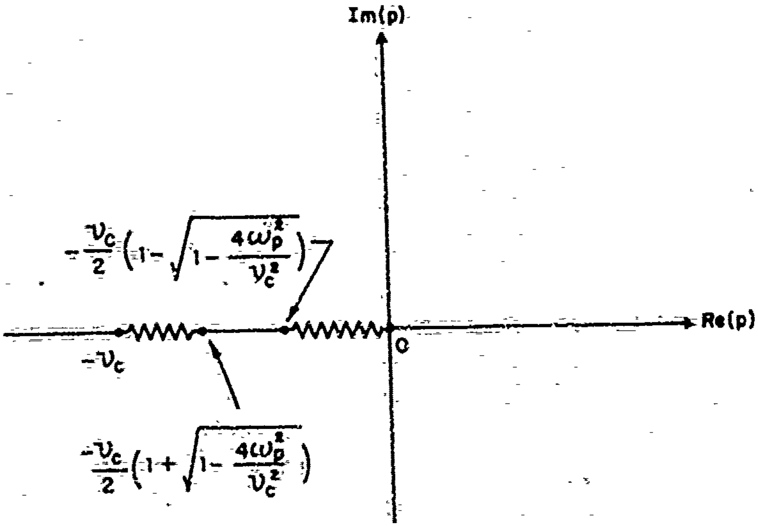

Figure 3. Branch Cuts for the Function γ When $\nu_c > 2\omega_p$

left-half p-plane. If we assume that $\hat{E}(x=0, p)$ has pole at s_o and s_1, then Eq. (1) can be rewritten

$$E(x,t) = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \hat{E}(x=0,p)\, e^{\left(pt-\frac{\gamma x}{c}\right)}\, dp \ , \tag{6}$$

where the contours C_1, C_2, C_3, and C_4 are shown in Figure 4.

To illustrate how the contour integrals are performed, let us consider the integral along C_1 in Figure 4. At a point A on C_1 we have $r_1 = \sigma$, $r_2 = \nu_c - \sigma$, $r_3 = r_4 = (\omega_p^2 - \nu_c\sigma + \sigma^2)^{1/2}$, and $\theta_1 = \pi$, $\theta_2 = 0$, $\theta_3 + \theta_4 = 0$. At the point B on C_1, r_1, r_2, r_3 and r_4 remain the same but θ_1 is now $-\pi$. Therefore at point A we find upon using these results in Eq. (5) that $\gamma = i[\sigma/(\nu_c-\sigma)]^{1/2}(\omega_p^2+\sigma^2-\nu_c\sigma)^{1/2}$, while at point B, $\gamma = -i[\sigma/(\nu_c-\sigma)]^{1/2}(\omega_p^2+\sigma^2-\nu_c\sigma)^{1/2}$. As a result the integral along contour C_1 can be written as [using the fact that $p = \sigma\exp(i\pi)$ and $dp = -d\sigma$]

$$\begin{aligned} I_{C_1} &= -\frac{1}{2\pi i}\int_0^{\nu_c} d\sigma\, \hat{E}(o, p=-\sigma)\, e^{-\sigma t} \exp\left[-i\frac{x}{c}\left(\frac{\sigma}{\nu_c-\sigma}\right)^{1/2}(\omega_p^2+\sigma^2-\nu_c\sigma)^{1/2}\right] \\ &\quad -\frac{1}{2\pi i}\int_{\nu_c}^{0} d\sigma\, \hat{E}(o, p=-\sigma)\, e^{-\sigma t} \exp\left[i\frac{x}{c}\left(\frac{\sigma}{\nu_c-\sigma}\right)^{1/2}(\omega_p^2+\sigma^2-\nu_c\sigma)^{1/2}\right] \\ &= \frac{1}{\pi}\int_0^{\nu_c} d\sigma\, \hat{E}(o,-\sigma)\, e^{-\sigma t} \sin\frac{x}{c}\left[\left(\frac{\sigma}{\nu_c-\sigma}\right)(\omega_p^2+\sigma^2-\nu_c\sigma)\right]^{1/2} \ . \end{aligned} \tag{7}$$

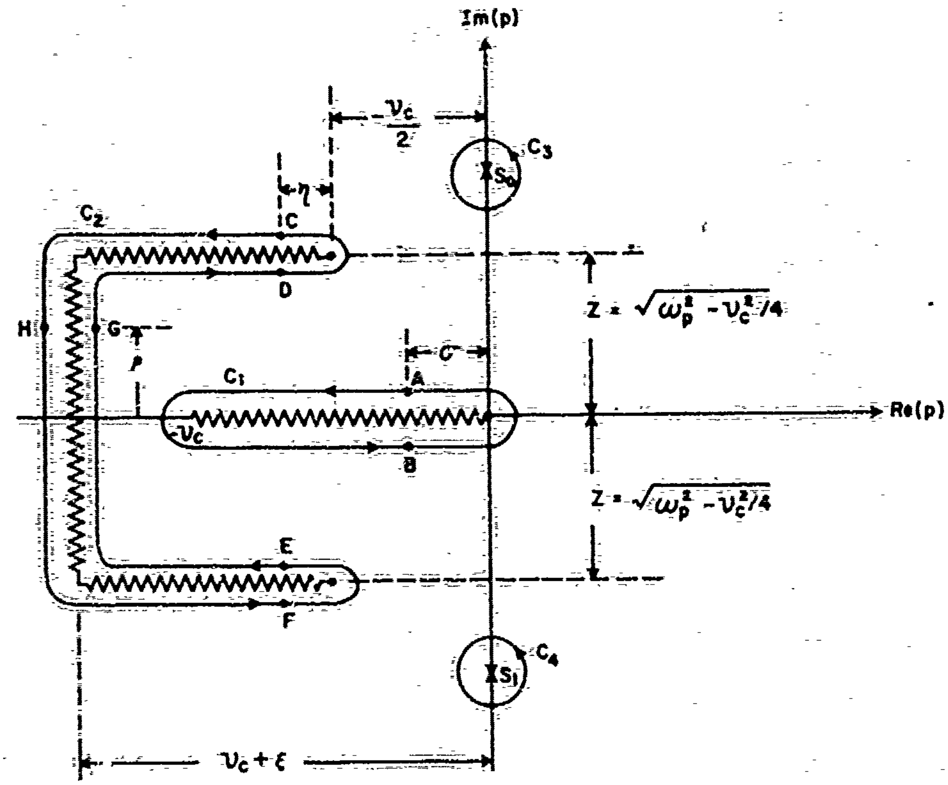

Figure 4. Deformed Contour of Integration for the Case When $\nu_c < 2\omega_p$

The integral along the contour C_2 is evaluated in the same fashion, while the integration along the contours C_3 and C_4 is trivial. For the interested reader, the details of the integration along C_2 are contained in the Appendix B to this report.

3. NUMERICAL RESULTS

3.1 Step-Carrier Sine Wave

We first consider the case when $E(x=0, t)$ is given by

$$E(x=0, t) = u(t) \sin \omega_o t \tag{8}$$

where $u(t)$ is the unit step function. For this case $\hat{E}(o, p) = \omega_o/(p^2+\omega_o^2)$, so that the poles s_o and s_1 shown in Figure 4 are $s_o = i\omega_o$, $s_1 = -i\omega_o$. If we define $\hat{r}_1 = \omega_o$, $\hat{r}_2 = (\omega_o^2+\nu_c^2)^{1/2}$, $\hat{r}_3 = [(\nu_c/2)^2 + (\omega_o - Z)^2]^{1/2}$, $\hat{r}_4 = [(\nu_c/2)^2 + (\omega_o+Z)^2]^{1/2}$ and

$\hat{\theta}_1 = \pi/2$, $\hat{\theta}_2 = \tan^{-1}(\omega_o/\nu_c)$, $\hat{\theta}_3 = \tan^{-1}[2(\omega_o - Z)/\nu_c]$ and $\hat{\theta}_4 = \tan^{-1}[2(\omega_o + Z)/\nu_c]$, we get

$$\gamma = \Gamma e^{i\Phi} = \left(\frac{\hat{r}_1\hat{r}_3\hat{r}_4}{\hat{r}_2}\right) \exp \frac{i}{2}\left[(\hat{\theta}_1 - \hat{\theta}_2 + \hat{\theta}_3 + \hat{\theta}_4)\right] \tag{9}$$

so that the integral over C_3 and C_4 yields

$$I_{C_3} + I_{C_4} = \exp(-\frac{x}{c}\Gamma\cos\Phi)\sin(\omega_o t - \Gamma\frac{x}{c}\sin\Phi) . \tag{10}$$

Using Eqs. (6), (7), (10) and (A5), we have calculated the time response of the electric field in the plasma for the step-carrier sine wave excitation of Eq. (8). Figure 5 shows the response at $\frac{\omega_o x}{c} = 1$ for an overdense plasma ($\omega_p/\omega_o = 2$) for different values of ν_c. We comment that the results for $\nu_c/\omega_p = 0.01$ are nearly identical with those obtained by Haskell and Case (1966a) for the case $\nu_c/\omega_p = 0$. For $\nu_c/\omega_p = 1$, however, the response is quite different. This difference is even more apparent for the cases when $\frac{\omega_o x}{c} = 5$ and $\frac{\omega_o x}{c} = 10$, which are shown in Figures 6 and 7. For large values of $\omega_o x/c$, even when $\nu_c/\omega_p = 0.1$, the transient

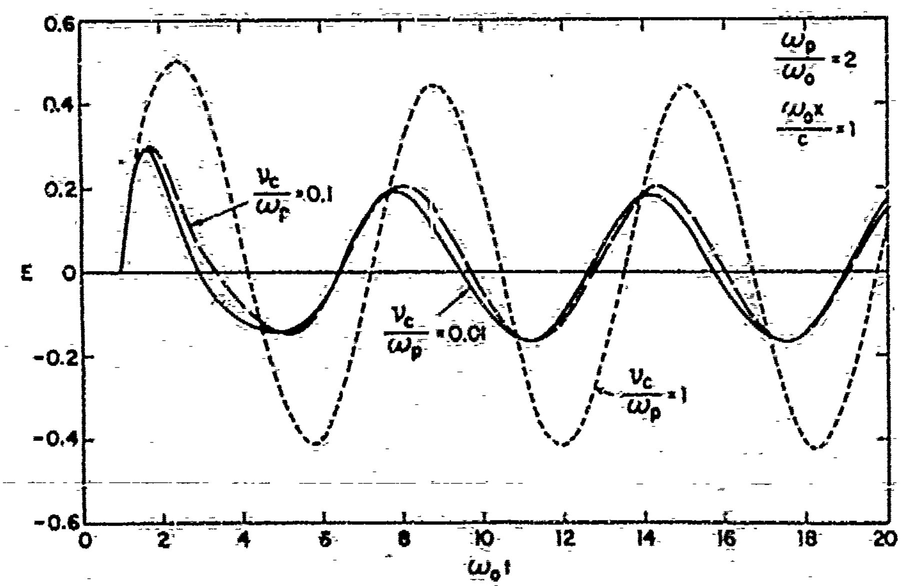

Figure 5. Transient Response to Step-Carrier Sine Wave for $\omega_o x/c = 1$ and $\omega_p/\omega_o = 2$

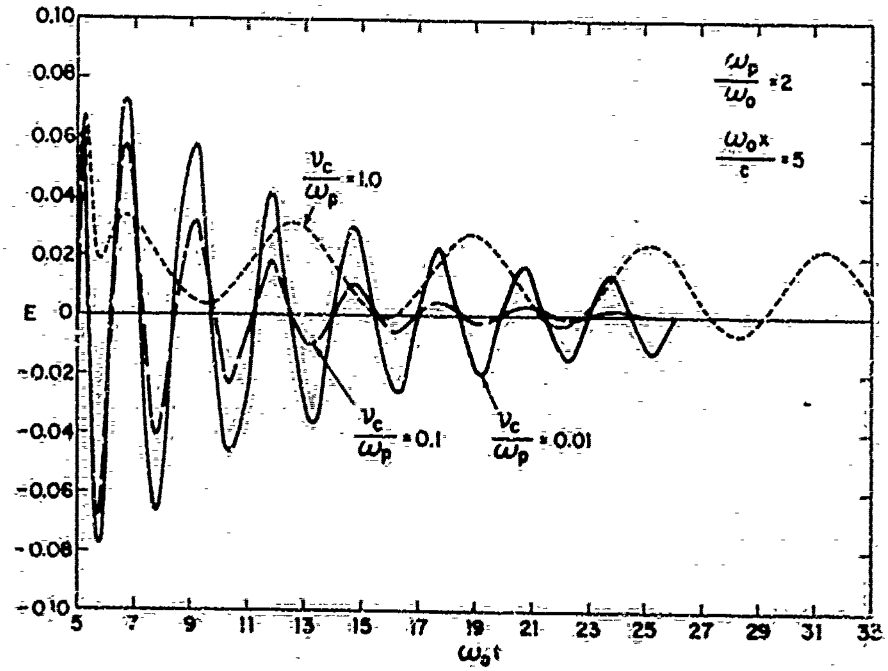

Figure 6. Transient Response to Step-Carrier Sine Wave for $\omega_o x/c = 5$ and $\omega_p/\omega_o = 2$

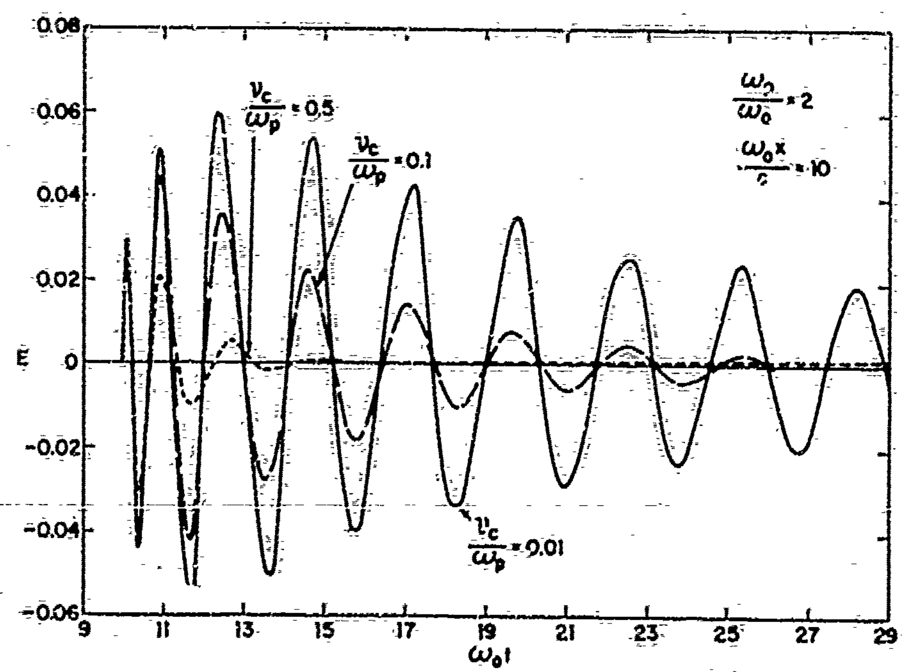

Figure 7. Transient Response to Step-Carrier Sine Wave for $\omega_o x/c = 10$ and $\omega_p/\omega_o = 2$

response is altered significantly, and for $\nu_c/\omega_p = 1$ the character of the response is completely different from the collisionless limit. Figures 8 and 9 illustrate how the nature of the transmitted signal changes as the distance into the plasma is varied. We note from Figures 8 and 9 that for large $\omega_o x/c$, there is an oscillation followed by a long wake in which the amplitude of the signal is nearly constant. It is readily shown that in this wake region, $E(x, t)$ can be approximated* by (see Appendix B)

$$E(x,t) \quad \exp\left(-\Gamma\frac{x}{c}\cos\Phi\right)\sin\left(\omega_o t - \Gamma\frac{x}{c}\sin\Phi\right) +$$

$$+ \frac{\omega_p}{2\omega_o c(\nu_c\pi)^{1/2}}\left(\frac{x}{t^{3/2}}\right)\exp\left(-\frac{\omega_p^2 x^2}{4c^2\nu_c t}\right). \tag{11}$$

In deriving Eq. (11) we have assumed that $t \gg x/c$ and that ν_c/ω_p is of order unity. From Eq. (11) we see that for large t the transient signal consists of a sinusoidal wave at frequency ω_o plus a component which eventually decays towards zero as $t^{-3/2}$.

Figures 10 through 12 indicate the nature of the response for $\omega_p/\omega_o = 4$. Figure 10 shows the transient response for $\omega_o x/c = 0.5$, where the effect of large values of ν_c is to radically alter the signal amplitude. From Figure 11 we see that when $\omega_o x/c = 2.5$, both the amplitude and the character of the transient response are radically changed from the case of zero collision frequency. In fact, the frequency of the oscillation for $\nu_c/\omega_p = 1$ is quite drastically reduced from the case when $\nu_c/\omega_p = 0.01$. Figure 12 illustrates the transient response at $\omega_o x/c = 5$. At this distance, for $\nu_c/\omega_p = 1$, there is hardly any oscillation of the transient response, but rather a slow buildup, and then an eventual decay (not shown in the figure) in accordance with the second term in Eq. (11), since the first term in Eq. (11) is quite small for this case.

In Figure 13, we have shown the effect of collisions on propagation in an underdense plasma. Here we note that no interesting or unexpected behavior occurs, and we will not study the underdense case further. In addition, this case has been previously considered by Case and Haskell (1967). We only comment that our results do agree with the previous ones obtained by Haskell and Case (1966a) in the limit $\nu_c/\omega_p \to 0$.

---

*To check the accuracy of Eq. (11), let us consider the case when $\nu_c/\omega_p = 1$, $\omega_p/\omega_o = 2$ and $\omega_o L/c = 10$. If we take a point in the signal wake at $\omega_o t = 15$, we have from Eq. (11) that $E = 2.54 \times 10^{-3}$, while the exact calculation (see Figure 9) gives $E = 2.51 \times 10^{-3}$.

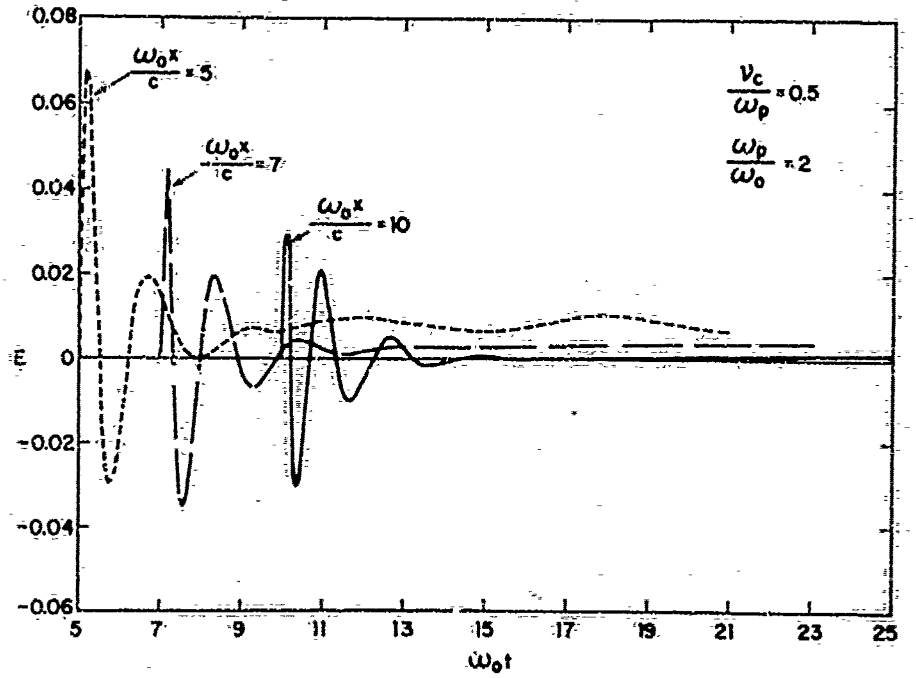

Figure 8. Transient Response to Step-Carrier Sine Wave for $\nu_c/\omega_p = 0.5$ and $\omega_p/\omega_o = 2$

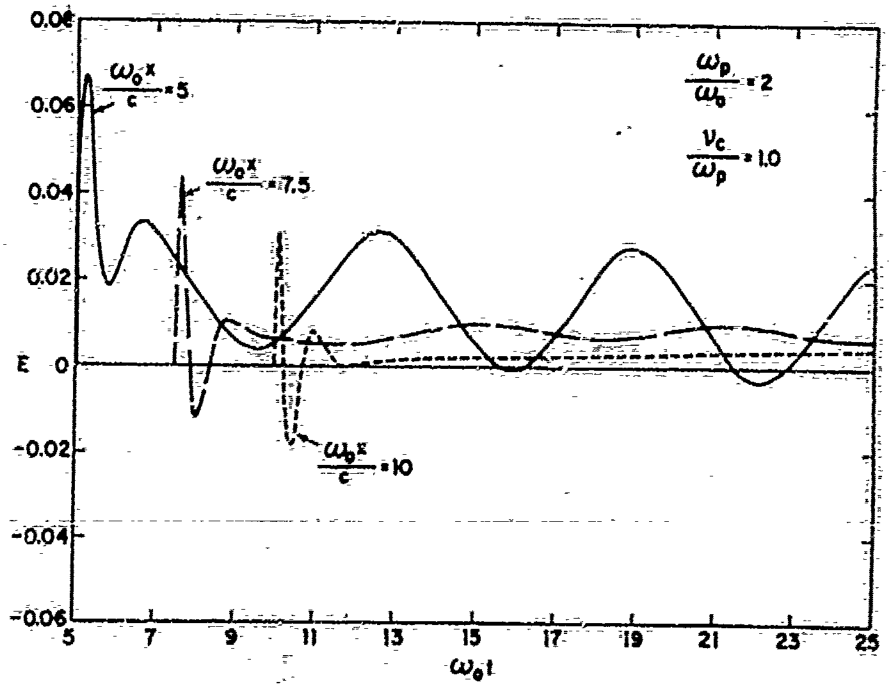

Figure 9. Transient Response to Step-Carrier Sine Wave for $\nu_c/\omega_p = 1.0$ and $\omega_p/\omega_o = 2$

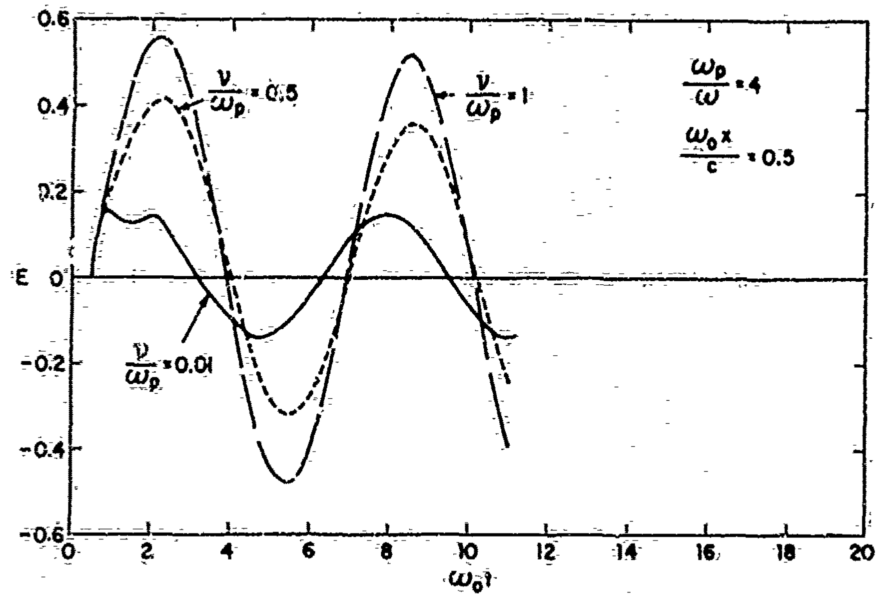

Figure 10. Transient Response to Step-Carrier Sine Wave for $\omega_0 x/c = 0.5$ and $\omega_p/\omega_0 = 4$

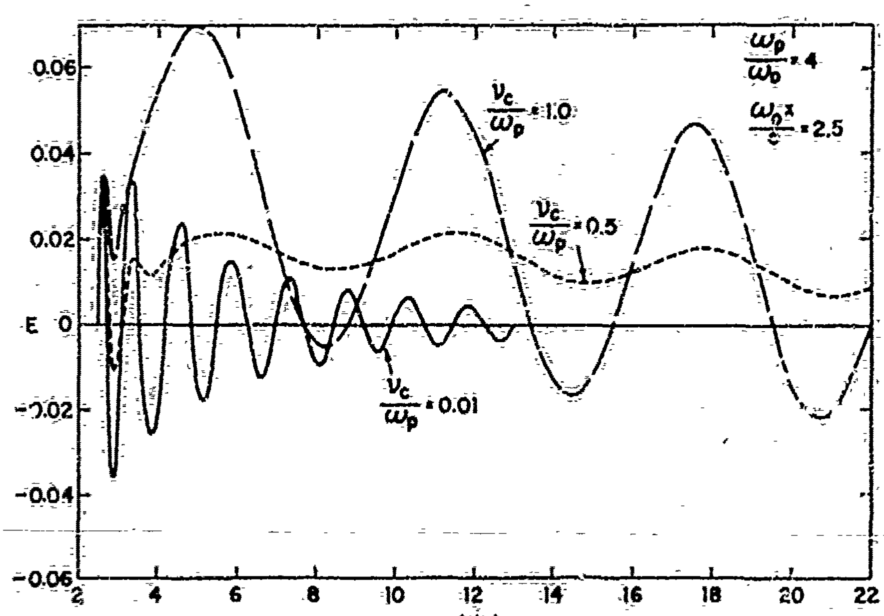

Figure 11. Transient Response to Step-Carrier Sine Wave for $\omega_0 x/c = 2.5$ and $\omega_p/\omega_0 = 4$

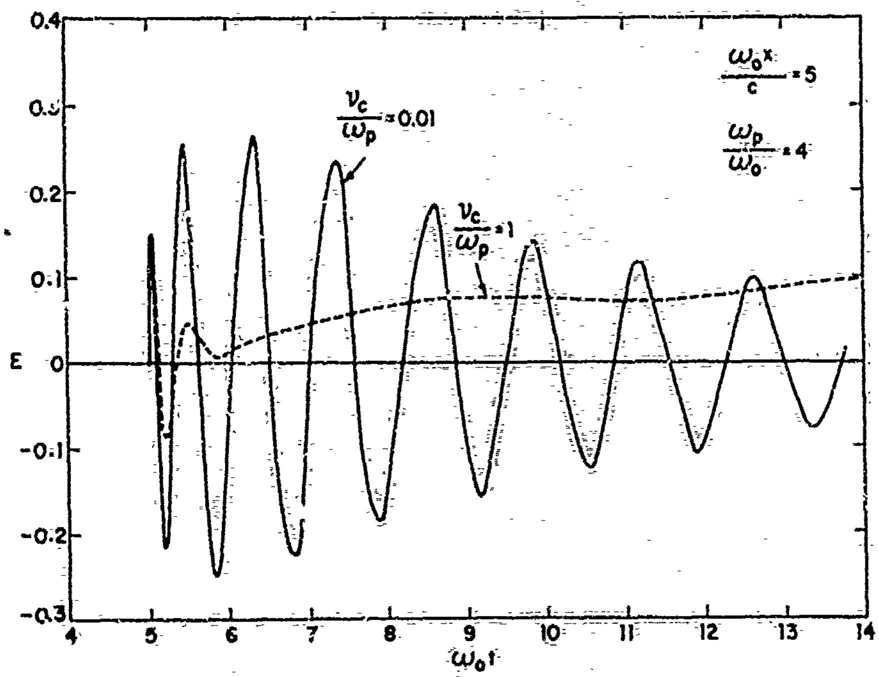

Figure 12. Transient Response to Step-Carrier Sine Wave for $\omega_o x/c$ = 5 and ω_p/ω_o = 4

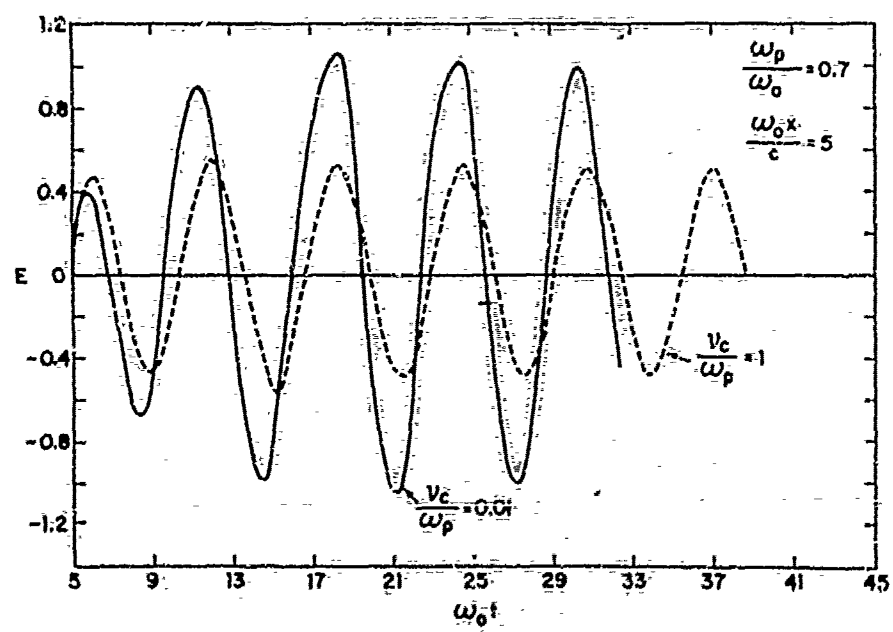

Figure 13. Transient Response to Step-Carrier Sine Wave for $\omega_o x/c$ = 5 and ω_p/ω_o = 0.7

3.2 Step Function

We next consider the situation when $E(x=0, t)$ is given by

$$E(x=0, t) = u(t) \quad . \tag{12}$$

For this case the pole s_0 coincides with the branch point at $p = 0$, while there is no pole s_1. Here, special care must be taken in performing the integral I_{C_1}, and the integral in Eq. (7) becomes

$$I_{C_1} = I_\delta + \frac{1}{\pi} \int_\delta^{\nu_c} d\sigma \hat{E}(0, -\delta)\, e^{-\sigma t} \sin\frac{x}{c}\left[\left(\frac{\sigma}{\nu_c - \sigma}\right)\left(\omega_p^2 + \sigma^2 - \nu_c\sigma\right)\right]^{1/2} , \tag{13}$$

where I_δ is the integral along a circle of radius δ surrounding the origin in the p-plane. That is

$$I_\delta = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, e^{\delta t \exp(i\theta)}\, e^{-\frac{x}{c}\gamma(p=\delta \exp(i\theta))} \quad . \tag{14}$$

Using the result of Eq. (13), plus the appropriate expression for I_{C_2} from Appendix A, we have calculated

$$E(x, t) = \int_{C_1} + \int_{C_2} \hat{E}(x=0, p)\, e^{(pt - \frac{x}{c}\gamma)}\, dp \; . \tag{15}$$

Figures 14 and 15 show the transient responses to the unit step input at distances $\omega_p x/c = 1$ and 5 within the plasma for $\nu_c/\omega_p = 0.01$, 0.1, and 1.0. It is interesting and important to note, from Figure 15, that when $\nu_c/\omega_p = 0.01$ the transient response for $\omega_p t < 40$ is nearly identical to the results obtained by Haskell and Case (1966a) for $\nu_c/\omega_p = 0$. However, we now note that when $\nu_c/\omega_p \neq 0$ the transient response does not approach zero as $t \to \infty$ as it does for $\nu_c/\omega_p = 0$, but rather approaches unity. It is clear from Figures 14 and 15 that the rate at which the transient grows towards unity is highly dependent on the collision frequency. For example, if $\nu_c/\omega_p = 1$ we see from Figure 15 that the transient has reached a value of e^{-1} by $\omega_p t \simeq 16$; for $\nu_c/\omega_p = 0.1$ the e^{-1} value occurs at $\omega_p t \simeq 160$; while for $\nu_c/\omega_p = 0.01$ the signal has reached e^{-1} of its final value of unity at $\omega_p t \simeq 1600$. Figure 16 illustrates the transient response at the larger distance $\omega_p x/c = 10$. The character of the response is essentially the same as for $\omega_p x/c = 5$, except that the rate of growth is toward the asymptotic value and

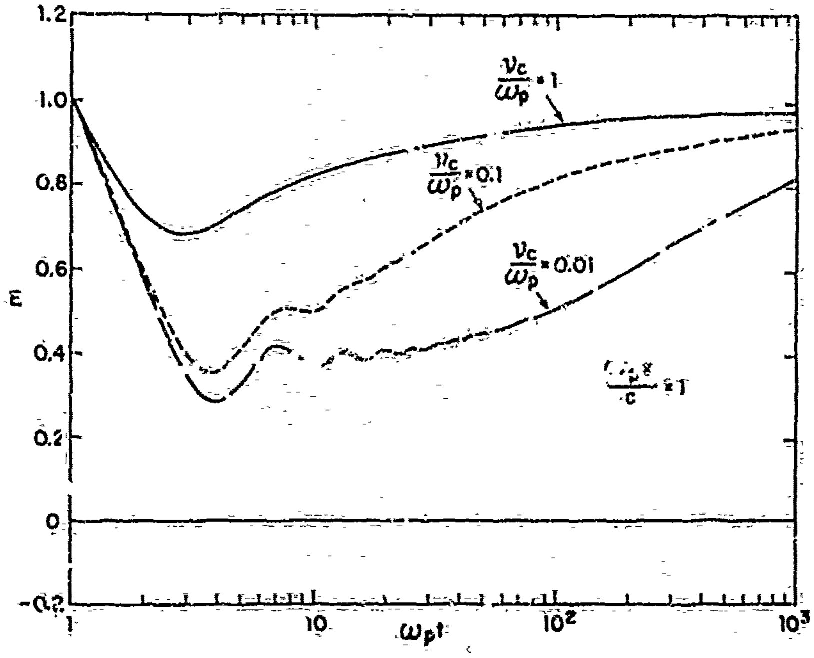

Figure 14. Transient Response to Unit Step for $\omega_p x/c = 1$

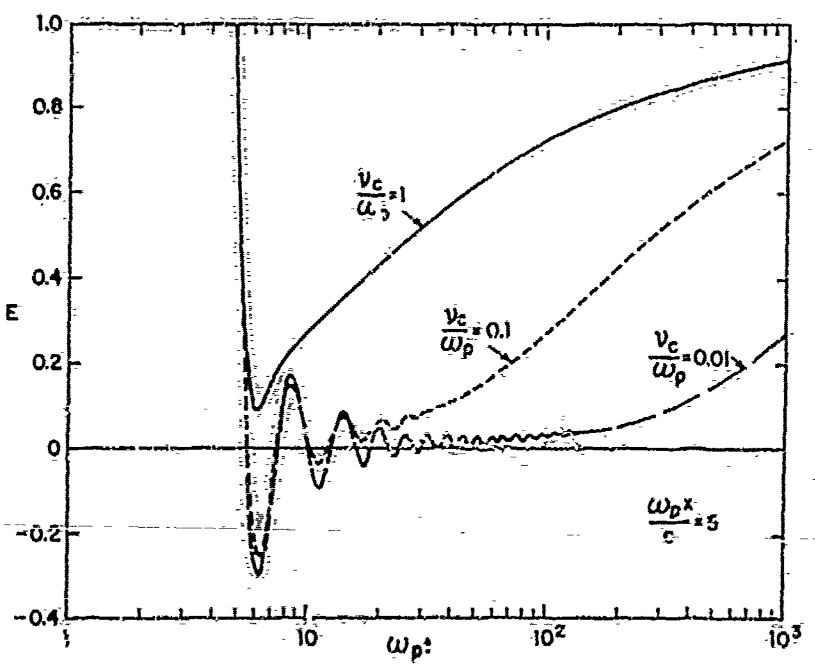

Figure 15. Transient Response to Unit Step for $\omega_p x/c = 5$

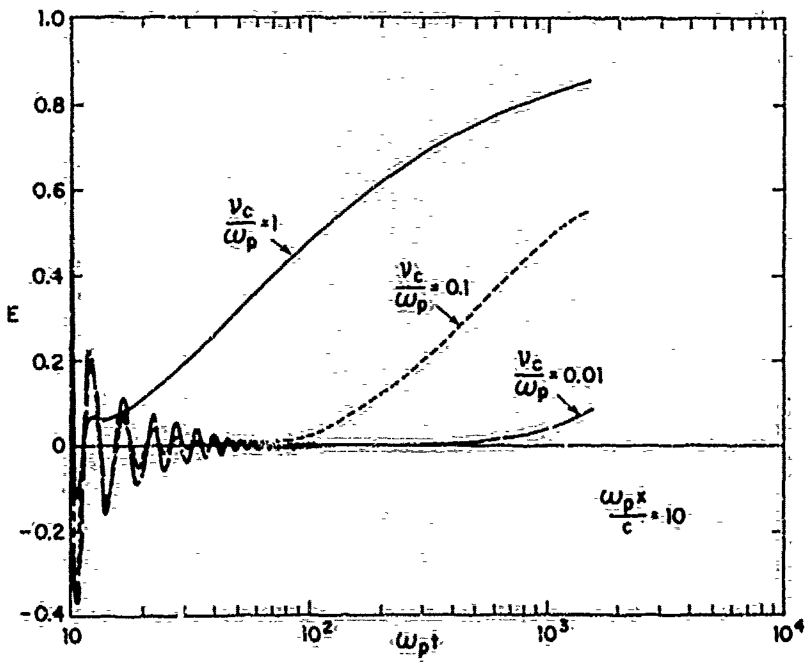

Figure 16. Transient Response to Unit Step for $\omega_p x/c = 10$

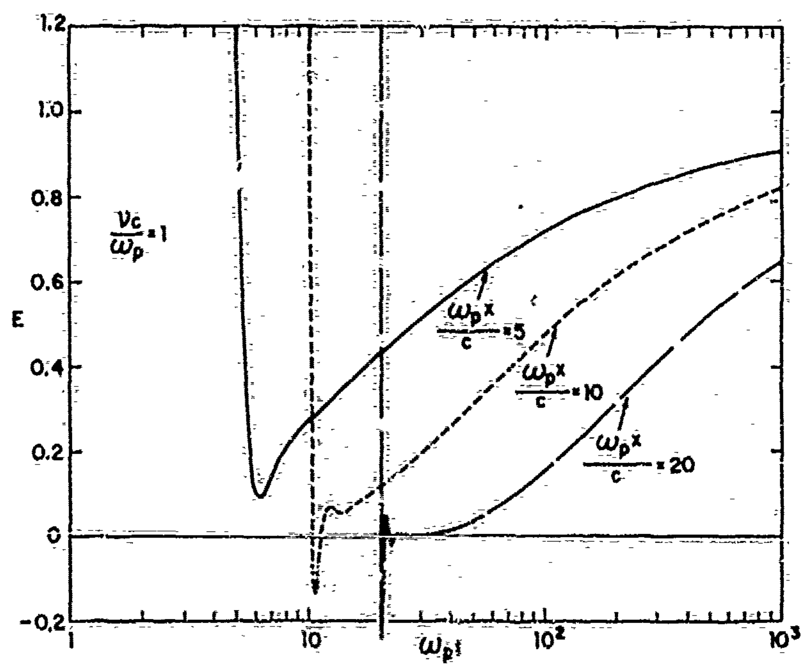

Figure 17. Transient Response to Unit Step for $\nu_c/\omega_p = 1$

is even slower than for $\omega_p x/c = 5$. The point is further illustrated in Figure 17, where we compare the responses for $\omega_p x/c = 5$, $\omega_p x/c = 10$ and $\omega_p x/c = 20$, for the case when $\nu_c/\omega_p = 1$.

The behavior exhibited in Figures 14 through 17 is just what is expected physically since a wave of frequency ω propagates as

$$\exp\left[i\frac{\omega}{c}x\left(1-\frac{\omega_p^2}{\omega(\omega+i\nu_c)}\right)^{1/2}\right].$$

The behavior near the leading edge of the responses is determined by the high frequency components which propagate as $\exp\left(i\frac{\omega x}{c}\right)$, while the time asymptotic behavior is determined by the low frequency components, which propagate as

$$\exp\left[x\left(\frac{i-1}{c}\right)\left(\frac{\omega_p^2\omega}{2\nu_c}\right)^{1/2}\right],$$

which approaches unity as $\omega \to o$.

It is possible to develop an accurate analytical expression for the field strength, which predicts the time asymptotic behavior we have observed in Figures 14 through 17. Let us assume that $t \gg x/c$, $\omega_p t \gg 1$, and ν_c/ω_p is of order unity. Then it is shown in Appendix C that

$$E(x,t) \to 1 - \mathrm{Erf}\left[\frac{\omega_p x}{2c(\nu_c t)^{1/2}}\right]. \tag{16}$$

This expression is quite accurate for $t \gg x/c$ and $\nu_c/\omega_p = o(1)$. To illustrate this we have compared the results of Eq. (16) with the exact result in Table 1, for $\nu_c/\omega_p = 1$ and $\omega_p x/c = 1$ and 10. We note that the agreement is excellent once we are not too close to $t = x/c$. Equation (16) is not nearly so accurate for smaller values of ν_c/ω_p since then it is no longer legal to neglect σ in comparison with ν_c in the argument of the sinc function in Eq. (13). A comparison of the exact results with the approximation of Eq. (16) for $\nu_c/\omega_p = 0.1$ is given in Table 2. As expected, the agreement is not generally good.

4. DISCUSSION

The most interesting point we have noted in this report is the effect of plasma losses on the transient response of a unit step. As pointed out in Section 1, this result has important consequences for the interaction of EMP with the plasma

Table 1. Comparison of Exact Results for the Transient E-Field With the Approximation of Eq. (16) for $\nu_c/\omega_p = 1$

$\frac{\omega_p x}{c} = 1$			$\frac{\omega_p x}{c} = 10$		
$\omega_p t$	Eq. (16)	Exact	$\omega_p t$	Eq. (16)	Exact
1	0.480	1.0	10	0.026	1.0
3	0.682	0.685	30	0.197	0.197
5	0.753	0.740	100	0.480	0.479
10	0.823	0.823	300	0.683	0.683
30	0.897	0.897	1000	0.823	0.823
100	0.944	0.944			
300	0.967	0.967			
1000	0.982	0.982			

Table 2. Comparison of Exact Results for the Transient E-Field With the Approximation of Eq. (16) for $\nu_c/\omega_p = 0.1$

$\frac{\omega_p x}{c} = 1$			$\frac{\omega_p x}{c} = 10$		
$\omega_p t$	Eq. (16)	Exact	$\omega_p t$	Eq. (16)	Exact
1	0.026	1.0	10	0.025	1.0
3	0.195	0.396	30	0.196	0.0062
5	0.315	0.410	100	0.480	0.0381
10	0.480	0.500	300	0.683	0.190
30	0.682	0.666	600	0.773	0.361
100	0.823	0.818	1000	0.823	0.479
300	0.897	0.896	1300	0.845	0.535
1000	0.944	0.943			

surrounding a reentry vehicle*. For example, suppose the plasma sheath thickness $x = 10$ cm, $\omega_p = 1.5 \times 10^{10}$ sec^{-1} and $\nu_c = 1.5 \times 10^{10}$ sec^{-1}. Then $\nu_c/\omega_p = 1$ and $\omega_p x/c = 5$. Now suppose we had a square electromagnetic pulse of duration $T_o = 10^{-6}$ sec incident upon the plasma sheath surrounding a reentry vehicle. Then from Figure 16 it is clear that the nature of the signal present at the skin of the reentry vehicle when $\nu_c = \omega_p$ is significantly different than for $\nu_c = 0$. This difference is illustrated qualitatively in Figure 18. It is possible using the results we have obtained for the unit step excitation to calculate the transient response for an actual EMP due to a nuclear blast, but in order to keep this report unclassified we have not presented this result here.

The calculation of the transient response presented here is readily extended to spatially inhomogeneous plasmas, by approximating the plasma by a series of homogeneous layers, each with different values of ω_p and ν_c. Then in the nth layer, $\hat{E}(\underline{x}, p)$ has the form

$$\hat{E}_n = A_n(p)\, e^{\gamma_n \frac{x}{c}} + B_n(p)\, e^{-\gamma_n \frac{x}{c}} \tag{17}$$

where A_n and B_n are obtained by requiring that E and $\partial E/\partial x$ be continuous across the interface between each layer, and γ_n is given by Eq. (3) with the values of ω_p and ν_c in the nth layer used.

The methods used here can also be readily extended to calculate transient behavior in a lossy dielectric or in a conductor. For this case, it is readily shown that the step function response at a point x within the material is given by

$$E(x, t) = 1 - \frac{1}{\pi} \int_0^1 \frac{dy}{y}\, e^{-y\tau} \sin\left[\ell y^{1/2}(1-y)^{1/2}\right], \tag{18}$$

where $\tau = (\sigma_o/\epsilon)t$, $\ell = \sigma_o x/(\epsilon v)$, $v = (\mu_o c)^{-1/2}$, ϵ is the dielectric permittivity of the material and σ_o it's conductivity. For $\tau \gg 1$ we obtain as an asymptotic representation for the step response

$$E(x, t) = 1 - \mathrm{Erf}\left(\frac{\ell}{2\tau^{1/2}}\right), \tag{19}$$

so that in a conductor or lossy dielectric, the unit step function response at a given point in the material eventually approaches unity for large values of τ. A plot of the transient response to a unit step in a lossy dielectric or conductor, as computed from Eq. 18, is shown in Figure 19.

*It is of little consequence for the EMP propagation in the ionosphere since there ν_c is so small that $\nu_c/\omega_p \ll 1$.

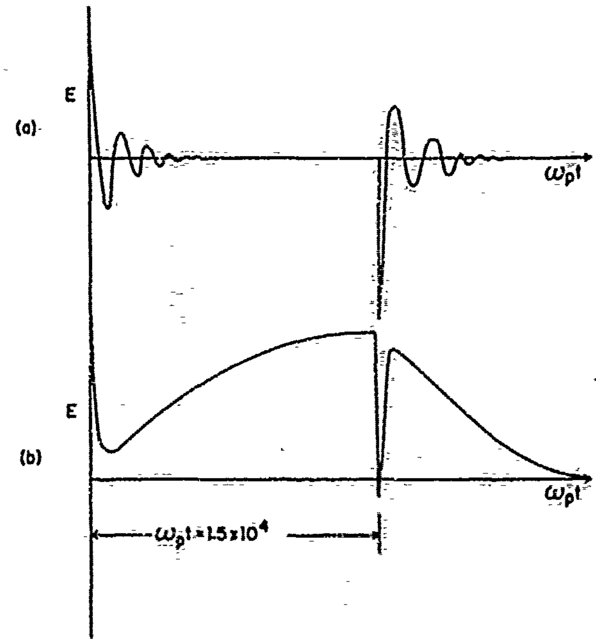

Figure 18. Response to a 10^{-6} sec Square Pulse for (a) $\nu_c = 0$, $\omega_p = 1.5 \times 10^{10}$ and (b) $\nu_c = \omega_p = 1.5 \times 10^{10}$

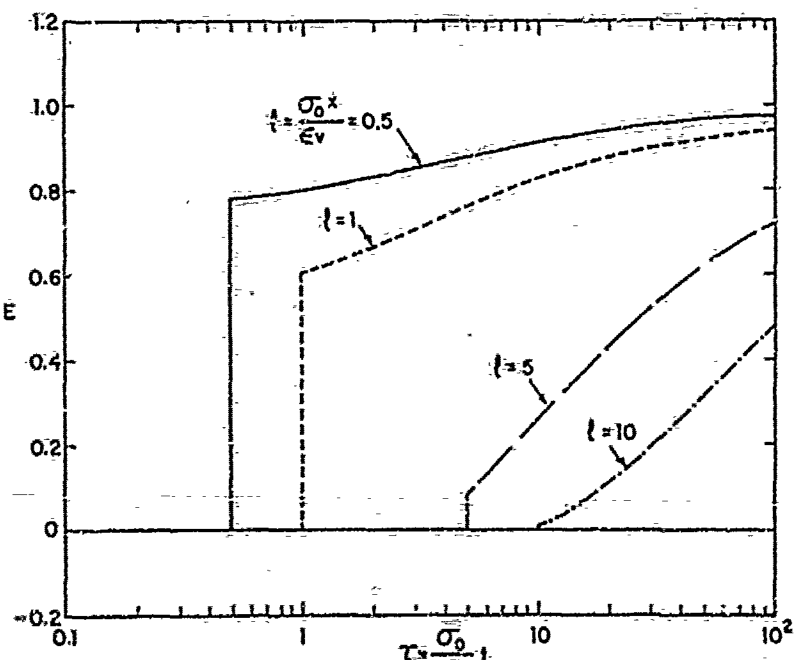

Figure 19. Transient Response of a Conductor of Lossy Dielectric to a Unit Step

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Appendix A

Here we discuss the integration along the contour C_2. Consider the point C shown in Figure 3. Here $r_1 \equiv \rho_1 = (\eta^2+\nu_c\eta+\omega_p^2)^{1/2}$, $r_2 \equiv \rho_2 = (\eta^2-\nu_c\eta+\omega_p^2)^{1/2}$, $r_3 \equiv \rho_3 = \eta$, $r_4 \equiv \rho_4 = (4\omega_p^2-\nu_c^2+\eta^2)^{1/2}$ and $\theta_1 \equiv \overline{\theta}_1 = \pi/2 + \tan^{-1}[(\nu_c/2+\eta)/Z]$, $\theta_2 \equiv \overline{\theta}_2 = \tan^{-1}[Z/(\nu_c/2-\eta)]$, $\theta_3 \equiv \overline{\theta}_3 = \pi$, $\theta_4 \equiv \overline{\theta}_4 = \pi/2 + \tan^{-1}(\eta/2Z)$. At the point D, all the above quantities are the same except θ_3 which now equals $-\pi$. Defining $\theta_c = 1/2\,(\overline{\theta}_1-\overline{\theta}_2+\overline{\theta}_3+\overline{\theta}_4)$ and $R = (\rho_1\rho_3\rho_4/\rho_2)^{1/2}$, we can then write the contour integral on the upper horizontal portion of C_2 as

$$I_2' = -\frac{1}{2\pi i}\int_0^{\frac{\nu_c}{2}+\xi} d\eta\, e^{-(\eta+\frac{\nu_c}{2}-iZ)t}\, e^{-\frac{x}{c}R(\cos\theta_c+i\sin\theta_c)}\,\hat{E}(p=-\frac{\nu_c}{2}-\eta+iZ)$$

$$+\frac{1}{2\pi i}\int_0^{\frac{\nu_c}{2}+\xi} d\eta\, e^{-(\eta+\frac{\nu_c}{2}+iZ)t}\, e^{\frac{x}{c}R(\cos\theta_c+i\sin\theta_c)}\,\hat{E}(p=-\frac{\nu_c}{2}-\eta+iZ). \qquad (A1)$$

The integrals over the lower horizontal portion of C_2 can be performed in a similar fashion, and yield

$$I_2'' = -\frac{1}{2\pi i}\int_0^{\frac{\nu_c}{2}+\xi} d\eta\, e^{-(\eta+\frac{\nu_c}{2}+iZ)t}\left[e^{\frac{x}{c}R(\cos\theta_c - i\sin\theta_c)}\right.$$

$$\left. - e^{-\frac{x}{c}R(\cos\theta_c - i\sin\theta_c)}\right]\hat{E}(p = -\frac{\nu_c}{2} - \eta - iZ) \quad . \tag{A2}$$

Combining Eqs. (A1) and (A2) we may write

$$I_2' + I_2'' = -\frac{e^{-\frac{\nu_c}{2}t}}{\pi}\left[\int_0^{\frac{\nu_c}{2}+\xi} d\eta\, e^{-\eta t} e^{-\frac{x}{c}R\cos\theta_c} I_m\left\{\hat{E}(p=-\frac{\nu_c}{2}-\eta+iZ)e^{i(Zt-\frac{x}{c}R\sin\theta_c)}\right\}\right.$$

$$\left. - \int_0^{\frac{\nu_c}{2}+\xi} d\eta\, e^{-\eta t} e^{\frac{x}{c}R\cos\theta_c} I_m\left\{\hat{E}(p=-\frac{\nu_c}{2}-\eta-iZ)\, e^{i(Zt+\frac{x}{c}R\sin\theta_c)}\right\}\right]. \tag{A3}$$

Finally we must perform the integrals over the vertical portions of C_2. Consider the point G. At this point $r_1 = \tilde{r}_1 = [(\nu_c+\xi)^2+\rho^2]^{1/2}$, $r_2 = \tilde{r}_2 \equiv (\xi^2+\rho^2)^{1/2}$, $r_3 = \tilde{r}_3 \equiv [(\nu_c/2+\xi)^2 + (Z-\rho)^2]^{1/2}$, $r_4 = \tilde{r}_4 \equiv [(\nu_c/2+\xi)^2 + (Z+\rho)^2]^{1/2}$, $\theta_1 = \tilde{\theta}_1 \equiv \pi - \tan^{-1}[\rho/(\nu_c+\xi)]$, $\theta_2 = \tilde{\theta}_2 \equiv \pi - \tan^{-1}(\rho/\xi)$, $\theta_3 = \tilde{\theta}_3 \equiv -\pi + \tan^{-1}[(Z-\rho)/(\frac{\nu_c}{2}+\xi)]$, $\theta_4 = \tilde{\theta}_4 = \pi - \tan^{-1}[(Z+\rho)/(\frac{\nu_c}{2}+\xi)]$.

At the point H, all the above quantities remain the same except $\theta_3 = \pi + \tan^{-1}[(Z-\rho)/(\frac{\nu_c}{2}+\xi)]$. Defining $\psi_o = \frac{1}{2}(\tilde{\theta}_1 - \tilde{\theta}_2 + \tilde{\theta}_3 + \tilde{\theta}_4)$ and $R_o = (\tilde{r}_1\tilde{r}_3\tilde{r}_4/\tilde{r}_2)^{1/2}$, the integrals on the vertical portion of C_2 can be written as

$$I_2''' = \frac{e^{-(\nu_c+\xi)t}}{\pi}\left[\int_0^Z d\rho\, e^{-\frac{x}{c}R_o\cos\psi_o} R_e\left\{\hat{E}(p=-\nu_c-\xi+i\rho)e^{i(\rho t-\frac{x}{c}R_o\sin\psi_o)}\right\}\right.$$

$$\left. - \int_0^Z d\rho\, e^{\frac{x}{c}R_o\cos\psi_o} Re\left\{\hat{E}(p=-\nu_c-\xi+i\rho)\, e^{i(\rho t+\frac{x}{c}R_o\sin\psi_o)}\right\}\right]. \tag{A4}$$

The total integral on the contour C_2 is then given by

$$I_{C_2} = I_2' + I_2'' + I_2''' . \tag{A5}$$

Appendix B

Here we outline the derivation of Eq. (11). We consider the limit when $t >> \frac{x}{c}$, $\omega_p t >> 1$ and ν_c/ω_p is of order unity. In this case it can be shown that the integral along the contour C_2 is negligible so that

$$E(x,t) \quad I_{C_1} + I_{C_3} + I_{C_4} , \tag{B1}$$

where I_{C_1} is given in Eq. (7) and $I_{C_3} + I_{C_4}$ is given in Eq. (10). Now in Eq. (7) it is clear that for t large, the principal contribution to the integral must come from σ near zero. Therefore I_{C_1} may be approximated by expanding the integrand in Taylor series about $\sigma = 0$. We get

$$I_{C_1} \simeq \frac{1}{\pi\omega_o} \int_0^{\nu_c} d\sigma\, e^{-\sigma t} \sin\left(\frac{\sigma}{\nu_c}\right)^{1/2} \frac{\omega_p x}{c}$$

$$\simeq \frac{1}{\pi\omega_o} \int_0^{\infty} d\sigma\, e^{-\sigma t} \sin\left[\frac{\omega_p x}{c}\left(\frac{\sigma}{\nu_c}\right)^{1/2}\right] . \tag{B2}$$

The integral in Eq. (B2) is a standard Laplace Transform and yields

$$I_{C_1} \simeq \frac{\omega_p}{2\omega_o c(\nu_c \pi)^{1/2}} \left(\frac{x}{t^{3/2}}\right) \exp\left(\frac{-\omega_p^2 x^2}{4c^2 \nu_c t}\right) . \tag{B3}$$

Finally, upon using Eqs. (B3) and (10) in Eq. (B1), we have

$$E(x,t) = \exp\left(-\Gamma \frac{x}{c} \cos \Phi\right) \sin\left(\omega_o t - \Gamma \frac{x}{c} \sin \Phi\right) + \\ + \frac{\omega_p}{2\omega_o c(\nu_c \pi)^{1/2}} \left(\frac{x}{t^{3/2}}\right) \exp\left(-\frac{\omega_p^2 x^2}{4c^2 \nu_c t}\right) . \tag{B4}$$

Appendix C

For a unit step when ν_c/ω_p is of order unity and $t \gg x/c$ we may neglect I_{C_2} so that the transient response is given by I_{C_1}, which from Eq. (13) with $\delta \to 0$ is

$$E(x,t) \cong 1 - \frac{1}{\pi}\int_0^{\lambda} \frac{dy}{y} e^{-uT} \sin L\left[\frac{y}{\lambda - y}\left(1 + y^2 - \lambda y\right)\right]^{1/2}, \tag{C1}$$

where $T = \omega_p t$, $\lambda = \nu_c/\omega_p$, $L = \omega_p x/c$. When $T \gg 1$, it is clear that the principal contribution to the integral comes from y near zero. Expanding the integrand about $y = 0$ and extending the range of integration to ∞ then yields

$$E(x,t) = 1 - \frac{1}{\pi}\int_0^{\infty} \frac{dy}{y} e^{-yT} \sin\left(\frac{L}{\lambda^{1/2}}\right) y^{1/2} . \tag{C2}$$

The integral in Eq. (B2) is a standard Laplace transform (Erdelyi et al, 1954) and upon evaluating it, we get

$$E(x,t) \cong 1 - \mathrm{Erf}\left[\frac{L}{2(\lambda T)^{1/2}}\right] . \tag{C3}$$

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Appendix D

Here we include a Fortran listing of the computer program used to calculate the transient response of a lossy homogeneous plasma. The program inputs are XL = $\omega_p x/c$, AL = ν_c/ω_p, OM = ω_o/ω_p, D = the radius δ of the contour in Eqs. (13) and (14) (D is usually set to 0.01), M = number of different values of T = $\omega_p t$ at which calculation is to be carried out, and L = 1 for a unit step input, while L = 2 for a step-carrier sine wave input. The program outputs are E E = E(x, t) = electric field strength, F F = contribution to E from poles, FU = contribution to E from integral along C_1, and FUNC - FUN = contribution to E from integral along C_2.

```
      PROGRAM RON(INPUT, OUTPUT)
      REAL  M2, M3, M4, N1, N2, N3, N4
      COMMON  AL, OM, XL, L, PI, ZO
      COMMON/TERMS/TERM1, TERM1SQ, TERM2, TERM2SQ, TERM3, TERM3SQ
      COMMON/D/D
    5 READ 10,  D, AL, OM, XL, L, M
      IF(AL .EQ. 0.0)  STOP
   10 FORMAT(4F10.4,2I4)
      PRINT 20,  AL, OM, XL, L
   20 FORMAT(1H1,49X,*AL = *F10.4/50X,*OM = *F10.4/50X,*XL = *F10.4/50X
     1,* L = *I5///9X,*T*,13X,*EE*,15X,*FF*,16X,*FU*,15X,*FUN*,15X,
     2*FUNC*,7X,*CP SECONDS*/)
      PI = 3.14159265358979
      TERM1 = 9.0 * AL
      TERM1SQ = TERM1**2
      TERM2 = 10.0 * AL
      TERM2SQ = TERM2**2
      TERM3 = 9.5 * AL
      TERM3SQ = TERM3**2
      ZO = SQRT(1.0 - 0.25*AL**2)
      F2 = SQRT(AL**2 + OM**2)
      M2 = AL / F2
      F3 = SQRT( (OM - ZO)**2 + (AL / 2.0)**2)
      M3 = 0.5*AL / F3
      F4 = SQRT( (OM + ZO)**2 + (AL / 2.0)**2)
      M4 = 0.5*AL / F4
      G = SQRT(OM) * SQRT(F3) * SQRT(F4) / SQRT(F2)
      P1 = 1.0 / SQRT(2.0)
      N1 = 1.0 / SQRT(2.0)
      P2 = SQRT(0.5*(1 - M2) )
      N2 = SQRT(0.5*(1 + M2) )
      N3 = SQRT(0.5*(1 + M3) )
      P3 = SQRT(0.5*(1 - M3) )
      IF(OM .LT. ZO)  P3 = -P3
      N4 = SQRT(0.5*(1 + M4) )
      P4 = SQRT(0.5*(1 - M4) )
      CP = N1*N2*N3*N4 + N1*N4*P2*P3 - N2*N4*P1*P3 + N3*N4*P1*P2
     1   - N2*N3*P1*P4 + N1*N3*P2*P4 - N1*N2*P3*P4 - P1*P2*P3*P4
      SP = P1*N2*N3*N4 - P2*N1*N3*N4 + P3*N1*N2*N4 + P1*P2*P3*N4
     1   + P4*N1*N2*N3 + N1*P2*P3*P4 - N2*P1*P3*P4 + N3*P1*P2*P4
      DO 30  J=1,M
      XX = FLOAT(J)
      IF(XX .LE. 6.0)  T = XL + 0.02*(XX - 1.0)
      IF(XX .GE. 7.0 .AND. XX .LE. 21.0)  T = XL + 0.1 + 0.2*(XX - 6.0)
      IF(XX .GE. 22.0)  T = XL + 3.1 + (XX - 21.0)
      IF(L .EQ. 1)  FF = FUNR(T)
      IF(L .EQ. 2)  FF = EXP(-G*XL*CP) * SIN(OM*T - G*XL*SP)
      FUNFU = FU(T)
      FUNFUN = FUN(T)
      FUNFUNC = FUNC(T)
      EE = FF + FUNFU - FUNFUN + FUNFUNC
      CPTIME = SECOND(A)
   30 PRINT 40,  T, EE, FF, FUNFU, FUNFUN, FUNFUNC, CPTIME
   40 FORMAT(5X,F8.3,1P,5(3X,E15.8),0P,3X,F9.3)
      GO TO 5
      END
*****************************************************************************
```

```
***********************************************************************
      FUNCTION FU(T)
      COMMON  AL, OM, XL, L, PI, ZO
      COMMON/TT1/TT1
      COMMON/D/D
      TT1 = T
      N = 1
      TWO = 0.0
      GO TO (5, 6), L
    5 H = (0.999 - D)*AL / 2.0
      FOUR = FUNK1(D*AL + H)
      ENDS = FUNK1(D*AL) + FUNK1(0.999*AL)
      GO TO 9
    6 H = 0.999*AL / 2.0
      FOUR = FUNK1(H)
      ENDS = FUNK1(0.00001) + FUNK1(0.999*AL)
    9 SUMO = (ENDS + 4.0*FOUR) * H / 3.0
   10 H = H / 2.0
      N = 2 * N
      TWO = TWO + FOUR
      FOUR = 0.0
      Y = H
      IF(L .EQ. 1)  Y = D*AL + H
      I = 0
   20 I = I + 1
      FOUR = FOUR + FUNK1(Y)
      Y = Y + H + H
      IF(I .LT. N)  GO TO 20
      FU = (ENDS + 2.0*TWO + 4.0*FOUR) * H / 3.0
      IF(ABS(SUMO - FU) .LT. 1.0E-6)  RETURN
      SUMO = FU
      GO TO 10
      END
***********************************************************************
***********************************************************************
      FUNCTION FUN(T)
      COMMON  AL, OM, XL, L, PI, ZO
      COMMON/TT2/TT2
      TT2 = T
      H = 9.5*AL / 2.0
      N = 1
      TWO = 0.0
      FOUR = FUNK2(H)
      ENDS = FUNK2(0.0) + FUNK2(9.5*AL)
      SUMO = (ENDS + 4.0*FOUR) * H / 3.0
   10 H = H / 2.0
      N = 2 * N
      TWO = TWO + FOUR
      FOUR = 0.0
      Y = H
      I = 0
   20 I = I + 1
      FOUR = FOUR + FUNK2(Y)
      Y = Y + H + H
      IF(I .LT. N)  GO TO 20
      FUN = (ENDS + 2.0*TWO + 4.0*FOUR) * H / 3.0
      IF(ABS(SUMO - FUN) .LT. 1.0E-6)  RETURN
      SUMO = FUN
      GO TO 10
      END
***********************************************************************
```

```
**********************************************************************
      FUNCTION FUNC(T)
      COMMON  AL, OM, XL, L, PI, ZO
      COMMON/TT3/TT3
      TT3 = T
      H = ZO / 2.0
      N = 1
      TWO = 0.0
      FOUR = FUNK3(H)
      ENDS = FUNK3(0.0) + FUNK3(ZO)
      SUMO = (ENDS + 4.0*FOUR) * H / 3.0
   10 H = H / 2.0
      N = 2 * N
      TWO = TWO + FOUR
      FOUR = 0.0
      Y = H
      I = 0
   20 I = I + 1
      FOUR = FOUR + FUNK3(Y)
      Y = Y + H + H
      IF(I .LT. N)  GO TO 20
      FUNC = (ENDS + 2.0*TWO + 4.0*FOUR) * H / 3.0
      IF(ABS(SUMO - FUNC) .LT. 1.0E-6)  RETURN
      SUMO = FUNC
      GO TO 10
      END
**********************************************************************
**********************************************************************
      FUNCTION FUNK1(Y)
      COMMON  AL, OM, XL, L, PI, ZO
      COMMON/TT1/TT1
      TERM = SIN(XL*SQRT(Y*(1.0 + Y**2 - AL*Y) / (AL - Y) ) )
      GO TO (10, 20), L
   10 FUNK1 = -EXP(-TT1*Y) * TERM / (PI * Y)
      RETURN
   20 FUNK1 = OM * EXP(-TT1*Y) * TERM / (PI * (Y**2 + OM**2) )
      RETURN
      END
**********************************************************************
**********************************************************************
      FUNCTION FUNK2(Y)
      COMMON  AL, OM, XL, L, PI, ZO
      COMMON/TT2/TT2
      AA = SQRT(SQRT(4.0 - AL**2 + Y**2) )
      AB = SQRT(SQRT(Y**2 + AL*Y + 1.0) )
      AC = SQRT(SQRT(Y**2 - AL*Y + 1.0) )
      R = SQRT(Y) * AA * AB / AC
      AO = ZO / SQRT(Y**2 + 1.0 + AL*Y)
      BO = (AL / 2.0 - Y) / SQRT(Y**2 + 1.0 - AL*Y)
      CO = 2.0*ZO / SQRT(Y**2 + 4.0 - AL**2)
      C1 = SQRT( ( 1.0 + AO ) / 2.0)
      C2 = SQRT( ( 1.0 + BO ) / 2.0)
      C3 = SQRT( ( 1.0 + CO ) / 2.0)
      S1 = SQRT( ( 1.0 - AO ) / 2.0)
      S2 = SQRT( ( 1.0 - BO ) / 2.0)
      S3 = SQRT( ( 1.0 - CO ) / 2.0)
      CT = -(C1*C2*C3 + C1*S2*S3 - C2*S1*S3 + C3*S1*S2)
      ST = -(S1*C2*C3 - S2*C1*C3 + S3*C1*C2 + S1*S2*S3)
```

```
      SS3 = SIN(ZO*TT2 - XL*R*ST - PI)
      CC3 = COS(ZO*TT2 - XL*R*ST - PI)
      SS4 = SIN(ZO*TT2 + XL*R*ST - PI)
      CC4 = COS(ZO*TT2 + XL*R*ST - PI)
      SS1 = SIN(ZO*TT2 - XL*R*ST - PI/2.0)
      CC1 = COS(ZO*TT2 - XL*R*ST - PI/2.0)
      SS2 = SIN(ZO*TT2 + XL*R*ST - PI/2.0)
      CC2 = COS(ZO*TT2 + XL*R*ST - PI/2.0)
      EXP1 = EXP(-XL*R*CT)
      EXP2 = EXP( XL*R*CT)
      EXP3 = EXP(-AL*TT2/2.0 - Y*TT2)
      TERM1 = AL/2.0 + Y
      GO TO (10, 20), L
   10 FUNK2 = 1.0 / (PI * (1.0 + AL*Y + Y**2) ) * EXP3 *
     1(EXP1*(ZO*SS1 - TERM1*CC1) - EXP2*(ZO*SS2 - TERM1*CC2) )
      RETURN
   20 TERM2 = TERM1**2
      TERM3 = (ZO - OM)**2
      TERM4 = (ZO + OM)**2
      FUNK2 = OM / (PI*(TERM2 + TERM4)*(TERM2 + TERM3) )*EXP3*(EXP1*
     1  ((ZO**2 - OM**2 - TERM2)*SS3 - 2.0*ZO*TERM1*CC3) - EXP2 *
     2  ((ZO**2 - OM**2 - TERM2)*SS4 - 2.0*ZO*TERM1*CC4) )
      RETURN
      END
*************************************************************************
*************************************************************************
      FUNCTION FUNK3(Y)
      REAL  K1, K2, K3, K4
      COMMON  AL, OM, XL, L, PI, ZO
      COMMON/TT3/TT3
      COMMON/TERMS/TERM1, TERM1SQ, TERM2, TERM2SQ, TERM3, TERM3SQ
C     TERM1 = 9.0 * AL
C     TERM1SQ = TERM1**2
C     TERM2 = 10.0 * AL
C     TERM2SQ = TERM2**2
C     TERM3 = 9.5 * AL
C     TERM3SQ = TERM3**2
      YSQ = Y**2
      BA = SQRT(SQRT((TERM2SQ + YSQ) / (TERM1SQ + YSQ) ) )
      BB = SQRT(SQRT(TERM3SQ + (ZO - Y)**2) )
      BC = SQRT(SQRT(TERM3SQ + (ZO + Y)**2) )
      R = BA * BB * BC
      A2 = TERM2 / SQRT(TERM2SQ + YSQ)
      B2 = TERM1 / SQRT(TERM1SQ + YSQ)
      C2 = TERM3 / SQRT(TERM3SQ + (ZO - Y)**2)
      D2 = TERM3 / SQRT(TERM3SQ + (ZO + Y)**2)
      K1 = SQRT( (1.0 + A2) / 2.0)
      K2 = SQRT( (1.0 + B2) / 2.0)
      K3 = SQRT( (1.0 + C2) / 2.0)
      K4 = SQRT( (1.0 + D2) / 2.0)
      T1 = SQRT( (1.0 - A2) / 2.0)
      T2 = SQRT( (1.0 - B2) / 2.0)
      T3 = SQRT( (1.0 - C2) / 2.0)
      T4 = SQRT( (1.0 - D2) /2.0)
      CS =   K1*K2*K3*K4 - K1*K4*T2*T3 + K2*K4*T1*T3 + K3*K4*T1*T2
     1     - K3*K2*T1*T4 + K1*K3*T2*T4 + K1*K2*T3*T4 + T1*T2*T3*T4
      SS = - T1*K2*K3*K4 + T2*K1*K3*K4 + T3*K1*K2*K4 + T1*T2*T3*K4
     1     - K1*K2*K3*T4 + K1*T2*T3*T4 - K2*T1*T3*T4 - K3*T1*T2*T4
      PQ = EXP(-TERM2 * TT3) / PI
```

```
      EXP1 = EXP(-R*XL*CS)
      EXP2 = EXP( R*XL*CS)
      BETA = XL*R*SS
      GO TO (10, 20), L
   10 THETA = Y*TT3 - PI/2.0
      SINM = SIN(THETA - BETA)
      SINP = SIN(THETA + BETA)
      COSM = COS(THETA - BETA)
      COSP = COS(THETA + BETA)
      FUNK3 = PQ / (TERM2SQ + YSQ) * (EXP1*(Y*COSM + TERM2*SINM)
     1        - EXP2 * (Y*COSP + TERM2*SINP) )
      RETURN
   20 THETA = Y*TT3 - PI
      SINM = SIN(THETA - BETA)
      SINP = SIN(THETA + BETA)
      COSM = COS(THETA - BETA)
      COSP = COS(THETA + BETA)
      FUNK3 = OM*PQ / ((TERM2SQ + (Y + OM)**2)*(TERM2SQ + (Y - OM)**2))*
     1    (EXP1*((YSQ - OM**2 - TERM2SQ)*COSM + 2.0*Y*AL*SINM) -
     2     EXP2*((YSQ - OM**2 - TERM2SQ)*COSP + 2.0*Y*AL*SINP))
      RETURN
      END
C*********************************************************************
C*********************************************************************
      FUNCTION FUNR(T)
      COMMON AL, OM, XL, L, PI, ZO
      COMMON/TT4/TT4
      TT4 = T
      H = PI / 2.0
      N = 1
      TWO = 0.0
      FOUR = FUNP(H)
      ENDS = FUNP(0.0) + FUNP(PI)
      SUMO = (ENDS + 4.0*FOUR) * H / 3.0
   10 H = H / 2.0
      N = 2 * N
      TWO = TWO + FOUR
      FOUR = 0.0
      Y = H
      I = 0
   20 I = I + 1
      FOUR = FOUR + FUNP(Y)
      Y = Y + H + H
      IF(I .LT. N)  GO TO 20
      FUNR = (ENDS + 2.0*TWO + 4.0*FOUR) * H / 3.0
      IF(ABS(SUMO - FUNR) .LT. 1.0E-5)   RETURN
      SUMO = FUNR
      GO TO 10
      END
C*********************************************************************
C*********************************************************************
      FUNCTION FUNP(Y)
      COMMON AL, OM, XL, L, PI, ZO
      COMMON/TT4/TT4
      COMMON/D/D
      SINE = SIN(Y)
      COSINE = COS(Y)
      RO2 = SQRT( (1.0 + D*COSINE)**2 + (D*SINE)**2)
      RO3 = SQRT( (ZO - D*AL*SINE)**2 + AL**2*(0.5 + D*COSINE)**2)
      RO4 = SQRT( (ZO + D*AL*SINE)**2 + AL**2*(0.5 + D*COSINE)**2)
```

```
      G1 = COSINE
      G2 = (1.0 + D*G1) / RO2
      G3 = AL * (0.5 + D*G1) / RO3
      G4 = AL * (0.5 + D*G1) / RO4
      RC1 = SQRT( (1.0 + G1) / 2.0)
      RC2 = SQRT( (1.0 + G2) / 2.0)
      RC3 = SQRT( (1.0 + G3) / 2.0)
      RC4 = SQRT( (1.0 + G4) / 2.0)
      RS1 = SQRT( (1.0 - G1) / 2.0)
      RS2 = SQRT( (1.0 - G2) / 2.0)
      RS3 = SQRT( (1.0 - G3) / 2.0)
      RS3 = - RS3
      RS4 = SQRT( (1.0 - G4) / 2.0)
      ACT = RC1*RC2*RC3*RC4 + RC1*RC4*RS2*RS3 - RC2*RC4*RS1*RS3 +
     1      RC3*RC4*RS1*RS2 - RS1*RS4*RC2*RC3 + RS2*RS4*RC1*RC3 -
     2      RS3*RS4*RC1*RC2 - RS1*RS2*RS3*RS4
      AST = RS1*RC2*RC3*RC4 - RS2*RC1*RC3*RC4 + RS3*RC1*RC2*RC4 +
     1      RS1*RS2*RS3*RC4 + RC1*RC2*RC3*RS4 + RC1*RS2*RS3*RS4 -
     2      RC2*RS1*RS3*RS4 + RC3*RS1*RS2*RS4
      ARR = SQRT(D * RO3 * RO4 / RO2) * XL
      ARC = ARR * ACT
      ARS = ARR * AST
      FUNP = (1.0/PI)*EXP(D*AL*TT4*COSINE - ARC)*COS(D*AL*TT4*SINE -ARS)
      RETURN
      END
```

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 03/16/73     CRL SCOPE 3.3     C03308A      02/01/72
16.25.56.TAYLXD1
16.25.56.TAYLX,CM10000,T10.
16.25.56.          7286     TAYLOR
16.25.56.NOBULL.
16.25.56.COPYSBF,INPUT,OUTPUT.
16.25.57.MASS STORAGE= 000062  PRUS
16.25.57.CP        .057 SEC.
16.25.57.PP        .518 SEC.
16.25.57.IO        .090 SEC.
```